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VARIATIONAL METHODS OF CONVOLUTION INTEGRAL AND OF LARGE SPRING CONSTANTS - A NUMERICAL COMPARISON

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Finite element solution formulations have been carried out for a simple initial value problem based on two different variational statements: that of convolutional integral developed by Gurtin and that of large spring constants adapted by this writer for initial value problems. Numerical results indicate that both generate convergent solution to the given initial value problem of a spring-mass system subjected to a harmonic forcing function.

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SUMMARY. Finite element solution formulations have been carried out for a simple initial value problem based on two different variational statements: that of convolutional integral developed by Gurtin and that of large spring constants adapted by this writer for initial value problems. Numerical results indicate that both generate convergent solution to the given initial value problem of a spring-mass system subjected to a harmonic forcing function.

1. INTRODUCTION. Through a simple initial value problem, this note demonstrates the use of the finite element discretization in conjunction with two variational formulations and compares the numerical results. The variational principles of convolutional integral for initial problems were developed by Gurtin some sixteen years ago (ref. 1 and 2). Since then these formulations have been applied to obtain solution of transient problems (ref. 3 and 4). However, the time dimension was treated separately from the spatial dimensions in the finite element approximation schemes. The viewpoint adopted in this note is that the separate treatment of spatial and time coordinates is unnecessary. Since the initial value problems are nonself-adjoint, the corresponding variational problems can be formulated with the help of adjoint field variables and thus can be used in Ritz-finite element solutions. One such formulation is used here to compare with the formulation using convolution integral in terms of numerical results for a simple initial value problem.

Let us consider a simple mass-spring system. The differential equation of the displacement u(t), a function of time t is

$$m\ddot{u} + ku = f_0 \cos \omega_f t \tag{1}$$

where m is the mass, k, the spring constant. A dot (·) denotes differentiation with respect to t. The parameters f_0 and w_f denote magnitude and frequency respectively, of the forcing function. The initial conditions are given as

$$u(0) = u_0 \quad , \quad u(0) = u_1 \tag{2}$$

We shall further use the equation

$$\omega^2 = \frac{k}{m} \quad , \quad f = \frac{f_o}{m}$$

Thus Eq. (1) has the form

$$\ddot{u} + \omega^2 u = f \cos \omega_f t \tag{3}$$

2. VARIATIONAL FORMULATION OF CONVOLUTIONAL INTEGRALS. The variational principle for the problem defined by Eqs. (3) and (2) is (ref. 2):

$$\delta I(u) = 0 \tag{4a}$$

where

$$I = \frac{1}{2}\,[u(t)*u(t) + \omega^2 t*u(t)*u(t)]$$
$$- [u_0 + \frac{f}{\omega_f^2} + u_1 t - \frac{f}{\omega_f^2} \cos \omega_f t]*u(t) \tag{4b}$$

The operator $*$ defines a convolution integral in the following equation

$$u(t)*v(t) = \int_o^t u(t-\tau)v(\tau)d\tau \tag{5}$$

where $u(t)$ and $v(t)$ are two arbitrary functions of t.

To see that the variational problem of Eqs. (4) is indeed equivalent to the original problem defined by Eqs. (3) and (2), one writes, from Eqs. (4):

$$\delta I = [u(t) + \omega^2 t*u(t) - (u_0 + \frac{f}{\omega_f^2} + u_1 t - \frac{f}{\omega_f^2} \cos \omega_f t)]*\delta u(t) = 0$$

for arbitrary $\delta u(t)$. Thus, $\delta I = 0$ leads to Eq. (6)

$$u(t) + \omega^2 t*u(t) - (u_0 + \frac{f}{\omega_f^2} + u_1 t - \frac{f}{\omega_f^2} \cos \omega_f t) = 0 \tag{6}$$

It is clear from Eq. (6) that $u(0) = u_0$.

Differentiate Eq. (6) once, one has

$$\dot{u}(t) + \omega^2 \int_o^t u(\tau)d\tau - u_1 - \frac{f}{\omega_f} \cos \omega_f t = 0 \tag{7}$$

Eq. (7) gives $\dot{u}(0) = u_1$. Thus both of the initial conditions are satisfied. The differential equation is recovered when Eq. (7) is differentiated once more. Note that in obtaining Eq. (7) the following differentiation formula has been used.

Let

$$F(t) = \int_o^t v(t-\tau)u(\tau)d\tau$$

Then

$$\frac{dF}{dt} = -\int_o^t \frac{\partial v}{\partial \tau}(t-\tau)u(\tau)d\tau + v(0)u(t)$$

3. VARIATIONAL FORMULATION WITH A LARGE "SPRING" CONSTANT. Consider the following variational problem

$$\delta I(u,v) = 0 \tag{8a}$$

with

$$I(u,v) = -\int_0^1 \dot{u}\dot{v}\,dt + \int_0^1 (\omega^2 u-f)v\,dt + \alpha[u(0) - u_0]v(1) - u_1 v(0)$$

In Eqs. (8), $u(t)$ is the physical field variable and $v(t)$ is the adjoint variable. This variational problem is unconstrained since the trial functions of neither $u(t)$ nor $v(t)$ are subject to any end condition requirements. To see that the set of Eqs. (8) is equivalent to the original initial value problem, it is only necessary to carry out the first variation and perform once integration-by-part. Thus, one has

$$\begin{aligned}\delta I(u,v) &= 0\\ &= \int_0^1 (\ddot{u}+\omega^2 u-f)\delta v\,dt\\ &+ \{\alpha[u(0) - u_0] - \dot{u}(1)\}\delta v(1) + [\dot{u}(0) - u_1]\delta v(0)\\ &+ \int_0^1 (\ddot{v}+\omega^2 v)\delta u\,dt\\ &+ \{\alpha\, v(1) + \dot{v}(0)\}\delta u(0)\ \dot{v}(1)\delta u(1)\end{aligned} \tag{9}$$

It is clear then if one chooses $v(t) = 0$ and let $\delta(t)$ be completely arbitrary, Eqs. (9) reduce to the original initial value problem as α approach to infinity.

4. PROCESS OF FINITE ELEMENT DISCRETIZATION. In case of convolutional integral, the variational equation used is Eq. (6) in Section 2. Rewrite Eq. (6) as

$$\delta I = [u(t) + \omega^2 t*u(t) - F(t)]*\delta u(t) = 0 \tag{10a}$$

where

$$F(t) = u_0 + \frac{f}{\omega_f^2} + u_1 t - \frac{f}{\omega_f^2} \cos \omega_f t \tag{10b}$$

In Eq. (10a), there are three convolution integrals to be evaluated: (a) $u(t)*\delta u(t)$; (b) $\omega^2 t*u(t)*\delta u(t)$; and (c) $F(t)*\delta u(t)$.

(a) For $u(t)*\delta u(t)$:

$$u(t)*\delta u(t) = \int_0^t u(t-\tau)du(\tau)d\tau$$

Let

$$\bar{\tau} = \tau/t$$

one has then

$$u(t)*\delta u(t) = t \int_0^1 \bar{u}(1-\bar{\tau})\delta\bar{u}(\bar{\tau})d\bar{\tau}$$

Consider

$$I = \int_0^1 \bar{u}(1-\bar{\tau})\delta\bar{u}(\bar{\tau})d\bar{\tau}$$

This integral is evaluated by finite element discretization.

$$I = \sum_{i=1}^{L} \int_{\ell_{i-1}}^{\ell_i} \bar{u}(1-\bar{\tau})\delta\bar{u}(\bar{\tau})d\bar{\tau}$$

Let

$$\ell_o = 0 \,, \quad \ell_L = 1 \,, \quad \ell_i = \frac{i}{L}$$

$$\xi = \xi^{(i)} = L\,\bar{\tau} - i + 1$$

$$\bar{\tau} = \frac{1}{L}\,[\xi + i - 1]$$

$$d\bar{\tau} = \frac{1}{L}\,d\xi$$

Hence

$$\bar{u}(\bar{\tau}) = \bar{u}[\frac{1}{L}\,(\xi+i-1)] = \bar{u}^{(i)}(\xi)$$

$$\bar{u}(1-\bar{\tau}) = \bar{u}[\frac{1}{L}\,(L-\xi-i+1)]$$

$$= \bar{u}[\frac{1}{L}\,(1-\xi+L-i+1)] = \bar{u}^{(L-i+1)}(1-\xi)$$

Thus

$$I = \sum_{i=1}^{L} \frac{1}{L} \int_0^1 \bar{u}^{(L-i+1)}(1-\xi)\delta\bar{u}^{(i)}(\xi)d\xi$$

Use the matrix representations for the shape function and generalized coordinates. One writes

$$\bar{u}^{(i)}(\xi) = \underset{\sim}{a}^T(\xi)\ \underset{\sim}{U}^{(i)}$$

and

$$\bar{u}^{(L-i+1)}(1-\xi) = a^T(1-\xi)U^{(L-i+1)}$$

Thus

$$I = \sum_{i=1}^{L} \frac{1}{L} \delta\underset{\sim}{U}^{(i)} \int_0^1 \underset{\sim}{a}(\xi)\underset{\sim}{a}^T(1-\xi)d\xi\ \underset{\sim}{U}^{(L-i+1)}$$

Or

$$I = \frac{1}{L} \sum_{i=1}^{L} \delta\underset{\sim}{U}^{(i)^T}\ \underset{\sim}{A}\ \underset{\sim}{U}^{(L-i+1)}$$

where

$$\underset{\sim}{A} = \int_0^1 \underset{\sim}{a}(\xi)\underset{\sim}{a}^T(1-\xi)d\xi \tag{11a}$$

Hence

$$\underset{\sim}{u}(t)*\delta u(t) = tI = \frac{t}{L} \sum_{i=1}^{L} \underset{\sim}{U}^{(i)} A\ \underset{\sim}{U}^{(L-i+1)} \tag{11b}$$

<u>(b) For $\omega^2 t*u(t)*\delta u(t)$:</u>

The evaluation of this double convolution integral is somewhat more complicated. First consider

$$t*u(t) = \int_o^t (t-\tau)u(\tau)d\tau$$

Then

$$\begin{aligned} I &= [t*u(t)]*\delta u(t) \\ &= \int_o^t \{\int_o^{t-\lambda} (t-\lambda-\tau)u(\tau)d\tau\}\delta u(\lambda)d\lambda \\ &= \int_o^t \int_o^{t-\lambda} (t-\lambda-\tau)u(\tau)\delta u(\lambda)d\tau d\lambda \end{aligned}$$

Again let

$$\bar{\tau} = \frac{\tau}{t} \quad , \quad \bar{\lambda} = \frac{\lambda}{t}$$

Thus $u(\tau)$ becomes $\bar{u}(\bar{\tau})$, $\delta u(\lambda)$ to $\delta\bar{u}(\bar{\lambda})$, etc. One has

$$I = t^3 \int_0^1 \int_0^{1-\bar{\lambda}} (1-\bar{\lambda}-\bar{\tau})\bar{u}(\bar{\tau})\delta\bar{u}(\bar{\lambda})\delta\bar{\tau}d\bar{\lambda} \tag{12}$$

It should be pointed out that the change of variables from τ,λ to $\bar{\tau},\bar{\lambda}$ (so that the limit of integration is changed from t to unit) is carried out after writing down explicitly the <u>double</u> convolutional integral and not before. This is due to the fact that the definition of a convolution integral requires that t appears explicitly in the integrals. To evaluate I of Eq. (12) we write

$$I = t^3\bar{I}$$

and work on $\bar{I}$ instead.

$$I = \int_0^1 \int_0^{1-\bar{\lambda}} (1-\bar{\lambda}-\bar{\tau})\bar{u}(\bar{\tau})\delta\bar{u}(\bar{\lambda})d\bar{\tau}d\bar{\lambda}$$

The area of integration in $(\bar{\lambda},\bar{\tau})$ plane is the triangle bounded by lines $\bar{\lambda} = 0$, $\bar{\tau} = 0$ and $\bar{\tau} = 1-\bar{\lambda}$ (shown in shaded area in Figure 1). Using the step function

$$H(1-\bar{\lambda}-\bar{\tau}) = \begin{cases} 1, & \bar{\tau} < 1 - \bar{\lambda} \\ 0, & \bar{\tau} > 1 - \bar{\lambda} \end{cases}$$

one can write

$$\bar{I} = \int_0^1 \int_0^1 H(1-\bar{\lambda}-\bar{\tau})(1-\bar{\lambda}-\bar{\tau})\bar{u}(\bar{\tau})\delta\bar{u}(\bar{\lambda})d\bar{\tau}d\bar{\lambda} \tag{13}$$

Equation (13) will be used for finite element discretization. We shall divide the unit square in $(\bar{\lambda},\bar{\tau})$ plane into smaller squares of L x L (Figure 2). Let

$$\begin{aligned} \xi &= \xi^{(i)} = L\bar{\lambda} - i + 1 \\ \eta &= \eta^{(i)} = L\bar{\tau} - j + 1 \end{aligned} \tag{14}$$

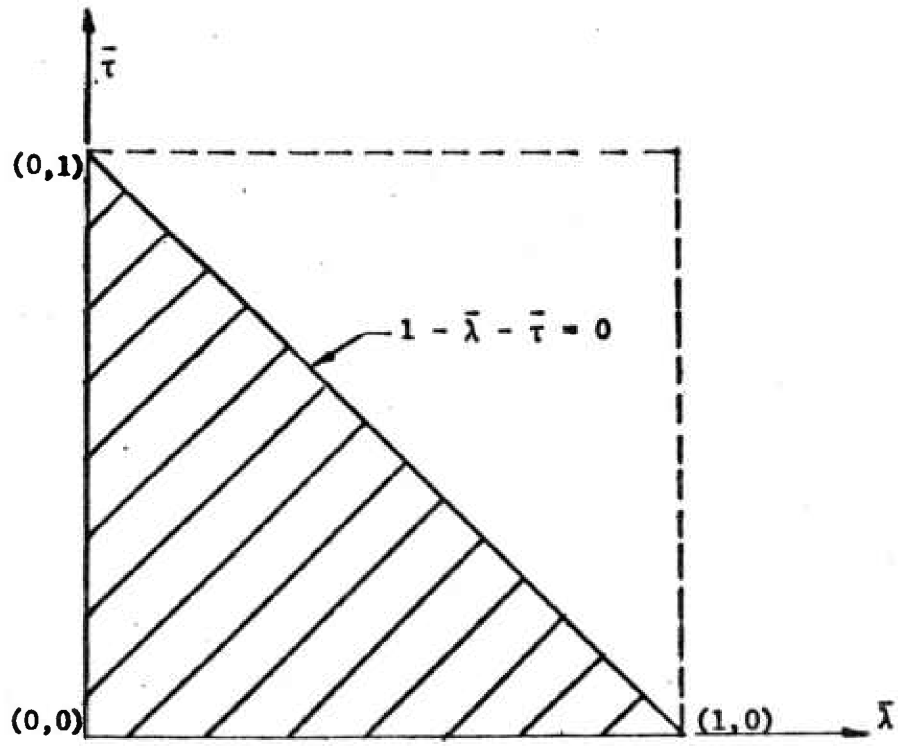

Figure 1. Area of integration for a double integral of convolution: $t*u(t)*\delta u(t)$.

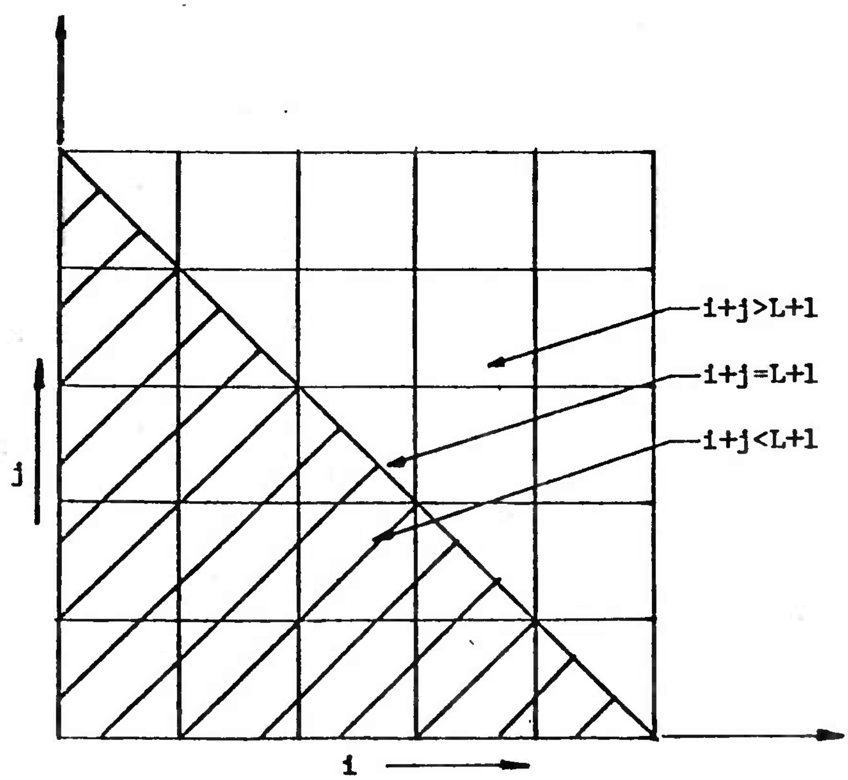

Figure 2. Area of integration using finite elements.

Thus

$$\delta\bar{y}(\bar{\lambda}) \rightarrow \delta\bar{y}^{(i)}(\xi)$$

$$\bar{y}(\bar{\tau}) \rightarrow \bar{y}^{(j)}(\eta)$$

$$1 - \bar{\lambda} - \bar{\tau} = \frac{1}{L}\{L + 1 - i - j + (1-\xi-\eta)\}$$

and

$$\bar{I} = \frac{1}{L^3} \sum_{i=1}^{L} \sum_{j=1}^{L} \int_0^1 \int_0^1 H^{(ij)}(L+1-i-j+1-\xi-\eta) \cdot$$

$$\cdot\ (L+1-i-j+1-\xi-\eta)\bar{y}^{(j)}(\eta)\delta y^{(i)}(\xi)d\eta d\xi \tag{15}$$

Or,

$$\bar{I} = \sum_{i=1}^{L} \sum_{j=1}^{L} I_{ij}/L^3 \tag{16}$$

with

$$I_{ij} = \int_0^1 \int_0^1 H^{(ij)}(L+1-i-j+1-\xi-\eta) \cdot$$

$$\cdot\ (L+1-i-j+1-\xi-\eta)\bar{y}^{(j)}(\eta)\delta y^{(i)}(\xi)d\xi d\eta \tag{17}$$

since $1 - \bar{\lambda} - \bar{\tau} = 0$, $L + 1 - i\ j + (1-\xi-\eta) = 0$.

Or,

$$1 - \xi - \eta = i + j - (L+1)$$

Thus, three cases to consider for $H^{(ij)}$

$$\begin{aligned} &(i) && H^{(ij)} = 1 \quad , \quad i + j < L + 1 \\ &(ii) && H^{(ij)} = 0 \quad , \quad i + j > L + 1 \\ &(iii) && H^{(ij)} = H(1-\xi-\eta) \quad , \quad i + j = L + 1 \end{aligned} \tag{18}$$

For case (i), one has

$$\bar{I}_{ij} = \int_0^1 \int_0^1 (L+2-i-j)\bar{y}^{(j)}(\xi)\delta\bar{y}^{(i)}(\xi)d\eta d\xi$$

$$= \delta \underset{\sim}{Y}^{(i)T} \int_0^1 \int_0^1 (L+2-i-j-\xi-\eta)\underset{\sim}{a}(\xi)\underset{\sim}{a}^T(\eta)d\eta d\xi\ \underset{\sim}{Y}^{(j)}$$

$$= \delta\underset{\sim}{Y}^{(i)}\ \bar{\underset{\sim}{A}}^{(ij)}\underset{\sim}{Y}^{(j)}$$

$$\bar{\underset{\sim}{A}}^{(ij)} = \int_0^1 \int_0^1 (L+2-i-j-\xi-\eta)\underset{\sim}{a}(\xi)\underset{\sim}{a}^T(\xi)d\eta d\xi$$

For case (ii),

$$I_{ij} = 0$$

For case (iii),

$$I_{ij} = \int_0^1 \int_0^1 H(1-\xi-\eta)(1-\xi-\eta)\bar{y}^{(j)}(\eta)\delta\bar{y}^{(i)}(\xi)d\eta d\xi$$

$$= \int_0^1 \int_0^{1-\xi} (1-\xi-\eta)\bar{y}^{(j)}(\eta)\delta\bar{y}^{(i)}(\xi)d\eta d\xi$$

$$= \delta\underset{\sim}{Y}^{(i)T} \int_0^1 \underset{\sim}{a}(\xi) \int_0^{1-\xi} (1-\xi-\eta)\underset{\sim}{a}^T(\eta)d\eta d\xi \; \underset{\sim}{Y}^{(j)}$$

$$= \delta\underset{\sim}{Y}^{(i)T} \bar{\bar{\underset{\sim}{A}}} \underset{\sim}{Y}^{(j)}$$

$$\bar{\bar{\underset{\sim}{A}}} = \int_0^1 \underset{\sim}{a}(\xi) \int_0^{1-\xi} (1-\xi-\eta)\underset{\sim}{a}^T(\eta)d\eta d\xi$$

Consequently,

$$I = t^3\bar{I}_1 = \frac{t^3}{L^3} \sum_{i=1}^{L} \sum_{j=1}^{L} I_{ij} \tag{19}$$

$$\begin{aligned} i + j < L + 1 &\rightarrow I_{ij} = \delta\underset{\sim}{Y}^{(i)}\underset{\sim}{A}^{(ij)}\underset{\sim}{Y}^{(j)} \\ i + j > L + 1 &\rightarrow I_{ij} = 0 \\ i + j = L + 1 &\rightarrow I_{ij} = \delta\underset{\sim}{Y}^{(i)}\bar{\bar{\underset{\sim}{A}}}\, \underset{\sim}{Y}^{(j)} \end{aligned} \tag{20}$$

$$\begin{aligned} \bar{\underset{\sim}{A}}^{(ij)} &= \int_0^1 \int_0^1 (L+2-i-j-\xi-\eta)\underset{\sim}{a}(\xi)\underset{\sim}{a}^T(\eta)d\xi d\eta \\ \bar{\bar{A}} &= \int_0^1 \underset{\sim}{a}(\xi) \int_0^{1-\xi} (1-\xi-\eta)\underset{\sim}{a}^T(\eta)d\eta d\xi \end{aligned} \tag{21}$$

And thus

$$\omega^2 t * u(t) * \delta u(t) = \omega^2 t^3 \; \bar{I} = \frac{\omega^2 t^3}{L^3} \sum_{i=1}^{L} \sum_{j=1}^{L} I_{ij} \tag{22}$$

(c) For $F(t)*\delta u(t)$ with $F(t)$ given in Eq. (10b), one has

$$F(t)*\delta u(t) = (a+bt+c \cos \omega_f t)*\delta u(t)$$

$$= a[1*\delta u(t)] + b[t*\delta u(t)] + c[\cos \omega_f t]*\delta u(t) \tag{23}$$

where, from Eq. (10b):

$$a = u_0 + \frac{f}{\omega_f^2} \quad , \quad b = u_1 \quad , \quad c = - \frac{f}{\omega_f^2} \tag{24}$$

Now, for Eq. (23), one has

$$1*\delta u(t) = \frac{t}{L} \sum_{i=1}^{L} \delta \underset{\sim}{U}^{(i)T} \int_0^1 \underset{\sim}{a}(\xi) d\xi \tag{25}$$

$$t*\delta u(t) = \frac{t^2}{L^2} \sum_{i=1}^{L} \delta \underset{\sim}{U}^{(i)T} \{(L-i+1) \int_0^1 \underset{\sim}{a}(\xi) d\xi - \int_0^1 \xi \underset{\sim}{a}(\xi) d\xi\} \tag{26}$$

and

$$\cos \omega_f t \; *\delta u(t) = \frac{t}{L} \sum_{i=1}^{L} \delta \underset{\sim}{U}^{(i)T} \int_0^1 \underset{\sim}{a}(\xi) \cos \left[\frac{\omega_f t}{L} (L-i+1-\xi)\right] d\xi \tag{27}$$

Now, Eqs. (11b), (22), (23) through (27), a global matrix equation can be written as

$$\delta \underset{\sim}{U}^T \underset{\sim}{K} \underset{\sim}{U} = \delta \underset{\sim}{U}^T \underset{\sim}{F} \tag{28}$$

Or

$$\underset{\sim}{K} \underset{\sim}{U} = \underset{\sim}{F} \tag{29}$$

which is then solved.

The finite element discretization procedure for the variational formulation using a large spring constant has been described elsewhere (see, for example, ref. 5) and will not be repeated here.

5. NUMERICAL RESULTS. Numerical values of the parameters in the given example as stated in Section 1 are as the following:

$$m = 1.0\ ,\quad k = 1.0\ ,\quad f_o = 1.0\ ,\quad \omega_f = 0.5$$

$$y_0 = 1.0\ ,\quad y_1 = 1.0$$

Computational results are presented in Tables 1 through 4. Table 1 and 2 compare results of the two methods in an interval of $0 \leq t \leq 10$, which is about the time for a complete forcing cycle. The results for $y(t)$ and $\dot{y}(t)$ are excellent for both methods. As the interval becomes shorter, $0 \leq t \leq 2$ as shown in Table 3 and 4, the convergence is further improved.

TABLE 1. NUMERICAL COMPARISONS BETWEEN TWO UNCONSTRAINED VARIATIONAL METHODS

$0 \leq t \leq 10.0$ 10 Elements

$y(t)$ / t	Convo Integ. M	Spring Const. M	Exact Solution
0	0.999	1.000	1.000
2.0	1.769	1.770	1.768
4.0	-1.094	-1.094	-1.094
6.0	-1.920	-1.920	-1.919
8.0	0.167	0.167	0.167
10.0	0.113	0.114	0.114

TABLE 2. NUMERICAL COMPARISONS BETWEEN TWO UNCONSTRAINED VARIATIONAL METHODS

$0 \leq t \leq 1.0$ 10 Elements

$y'(t)$ / t	Convo Integ. M	Spring Const. M	Exact Solution
0	1.011	1.004	1.000
2.0	-0.675	-0.675	-0.674
4.0	-1.520	-1.518	-1.512
6.0	0.780	0.778	0.773
8.0	0.691	0.690	0.689
10.0	-0.391	-0.385	-0.381

TABLE 3. NUMERICAL COMPARISONS BETWEEN TWO UNCONSTRAINED VARIATIONAL METHODS

$0 \leq t \leq 2.0$ 10 Elements

y(t) / t	Convo Integ. M	Spring Const. M	Exact Solution
0	1.000000	1.000000	1.000000
0.4	1.389154	1.389154	1.389153
0.8	1.713203	1.713203	1.713203
1.2	1.911703	1.911702	1.911701
1.6	1.938251	1.938251	1.938249
2.0	1.768413	1.768416	1.768416

TABLE 4. NUMERICAL COMPARISONS BETWEEN TWO UNCONSTRAINED VARIATIONAL METHODS

$0 \leq t \leq 2.0$ 10 Elements

y'(t) / t	Convo Integ. M	Spring Const. M	Exact Solution
0	0.99999	1.00000	1.00000
0.4	0.91844	0.91843	0.91842
0.8	0.67623	0.67622	0.67621
1.2	0.29662	0.29662	0.29661
1.6	-0.17425	-0.71424	-0.17425
2.0	-0.67425	-0.67413	-0.67403

In conclusion, we have observed that the numerical convergence of the method of large spring constants, in the simple example given, is at least as good as that of the formulation through the variational principle of convolutional integrals. Both are easily adapted for finite element discretization. Due to the fact that the variational principles of convolutional integrals can be formulated only for a very restricted class of problems (of constant coefficients, for example). The alternate approach of large spring constants appears to be quite attractive to obtain solutions of non-self-adjoint problems in general and of initial value problems and initial boundary value problems in particular.

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